



GESAR

The Group for Environmental
Studies in Reservoirs

EXERCÍCIO - MÉTODO DE ELEMENTOS FINITOS



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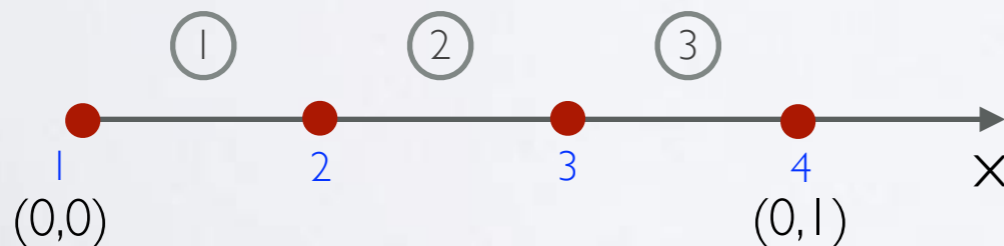
PROBLEMA 1D - FORMA FORTE

Encontrar u em $\Omega = [0, 1]$ tal que:

$$\frac{d^2 u}{dx^2} + u + 1 = 0 \quad \left| \quad \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{array} \right. \quad \leftarrow \quad \begin{array}{l} \text{condi\c{c}o} \\ \text{de contorno} \end{array}$$

domain: $h_1 = h_2 = h_3 = 1/3$

Solu\c{c}o: $u_2 = 1.049$;
 $u_3 = 1,874$;
 $u_4 = 2,386$



PROBLEMA 1D - FORMA FRACA

Encontrar u em H^1 tal que:

$$\int_{\Omega} w \left(\frac{d^2 u}{dx^2} + u + 1 \right) d\Omega = 0$$

função peso

procedimento matemático (integração por partes)

$$\int_0^1 w \frac{d^2 u}{dx^2} dx + \int_0^1 w u dx + \int_0^1 dx = 0$$

$$w \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

FUNÇÕES DO MEF

Propriedades do elemento finito

- as funções de forma assumem o valor unitário no nó designado e zero nos demais nós;
- a soma de todas as funções de forma em um elemento é igual a um em todo o elemento, incluindo o contorno.

Tabela

item	nó, i	nó, j	x arbitrário
N_i	1	0	entre 0 e 1
N_j	0	1	entre 0 e 1
$N_i + N_j$	1	1	1

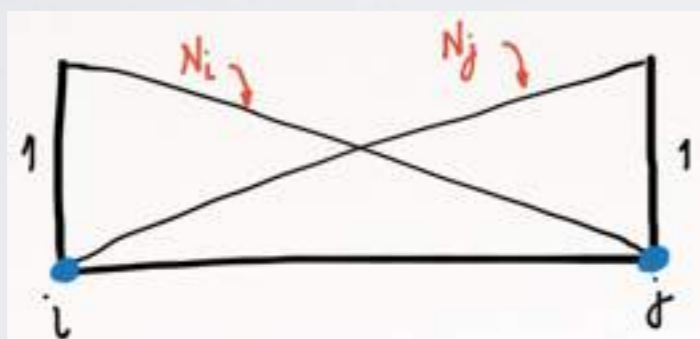
FUNÇÕES DO MEF

Problema 1D - **linear**:

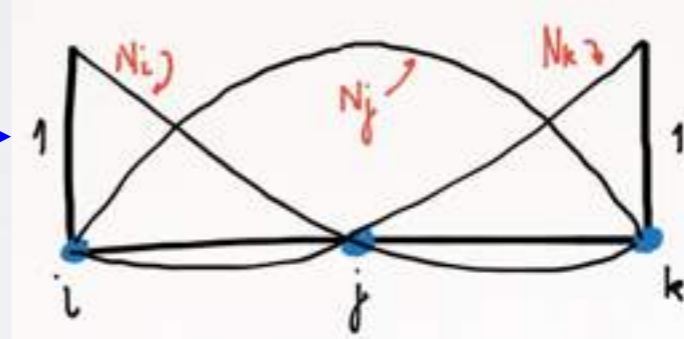
$$T(x) = \alpha_1 + \alpha_2 x$$

Problema 1D - **quadrático**:

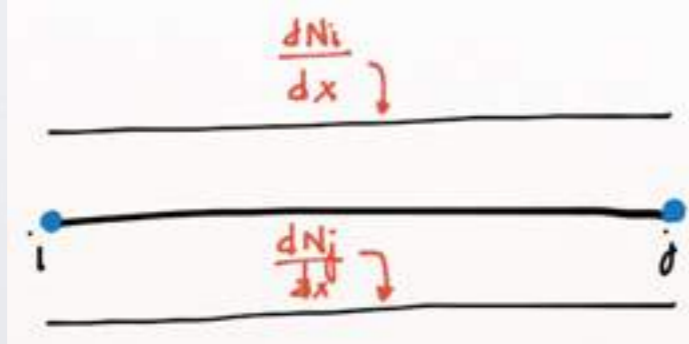
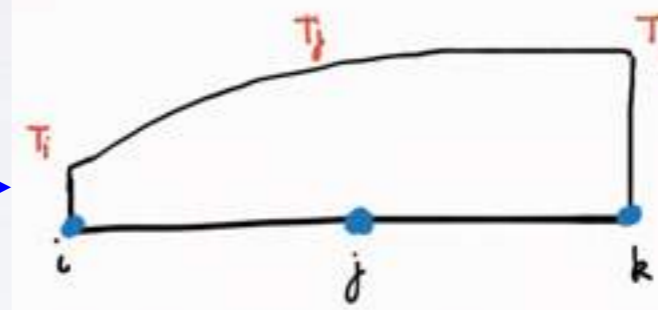
$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



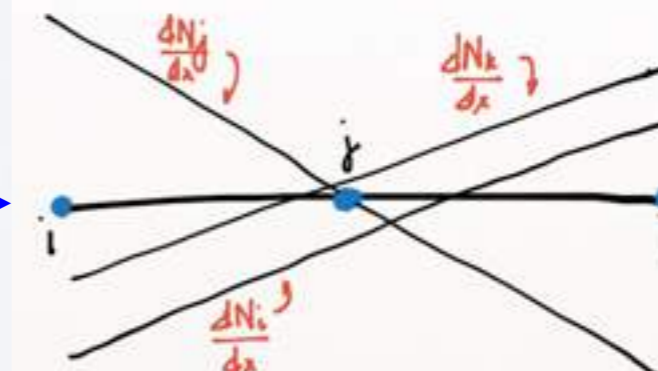
função de forma



variação de temperatura



derivada da função de forma



MÉTODO DE GALERKIN

funções aproximadoras: $\hat{u} = \sum_{i=1}^4 N_i u_i$ $\hat{w} = \sum_{j=1}^4 N_j w_j$

$$w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0) - \sum_{i,j=1}^4 \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx +$$

$$+ \sum_{i,j=1}^4 \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^4 \int_0^1 N_j w_j dx = 0$$

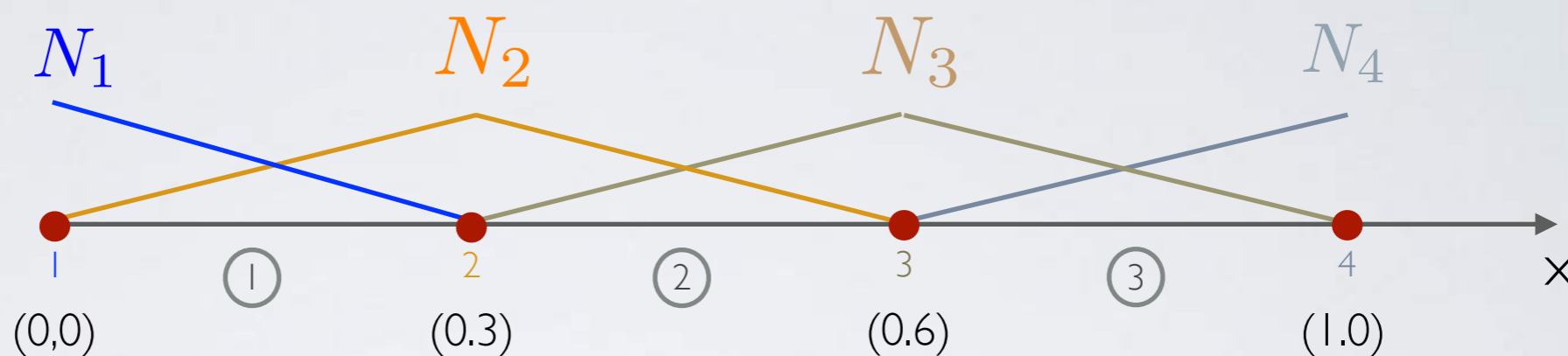
$$\sum_{i,j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) u_i = \sum_{j=1}^4 \int_0^1 N_j dx + \frac{du}{dx}(1) - \frac{du}{dx}(0)$$

matriz de rigidez K_{ij} matriz de massa M_{ij} vetor lado direito b_i condição de contorno

$$(K_{ij} - M_{ij}) u_i = b_i + b.c.$$

FUNÇÃO DE FORMA LINEAR

domínio
e funções
de forma



elemento ① $N_1 = -3x + 1$
 $\Omega_1^e = [0, 1/3]$ $N_2 = 3x$

elemento ② $N_2 = -3x + 2$
 $\Omega_2^e = [1/3, 2/3]$ $N_3 = 3x - 1$

elemento ③ $N_3 = -3x + 3$
 $\Omega_3^e = [2/3, 1]$ $N_4 = 3x - 2$

FORMA MATRICIAL

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_2 N_2 dx$$

matriz

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element $\textcircled{1}$

$$\Omega_1^e = [0, 1/3]$$

$$N_1 = -3x + 1$$

$$N_2 = 3x$$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vetor

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

FORMA MATRICIAL

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matriz

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element ②

$$\Omega_2^e = [1/3, 2/3]$$

$$N_2 = -3x + 2$$

$$N_3 = 3x - 1$$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vetor

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

FORMA MATRICIAL

$$K_{33} - M_{33} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_4 N_4 dx$$

matriz

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element ③

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

$$b_3 = \int_{2/3}^1 N_3 dx$$

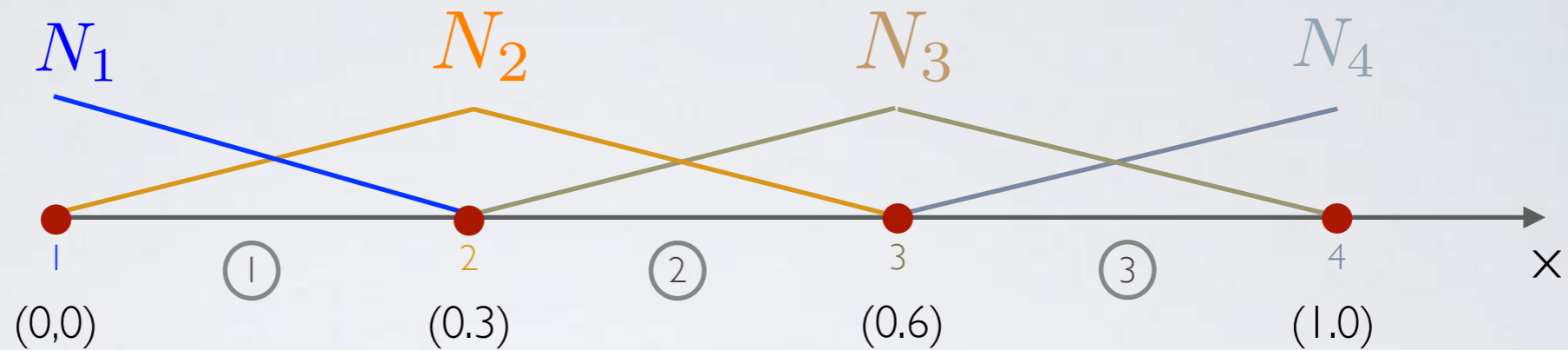
$$b_4 = \int_{2/3}^1 N_4 dx$$

vetor

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

FORMA MATRICIAL

domain
and shape
functions:



element $\textcircled{1}$
 $\Omega_1^e = [0, 1/3]$

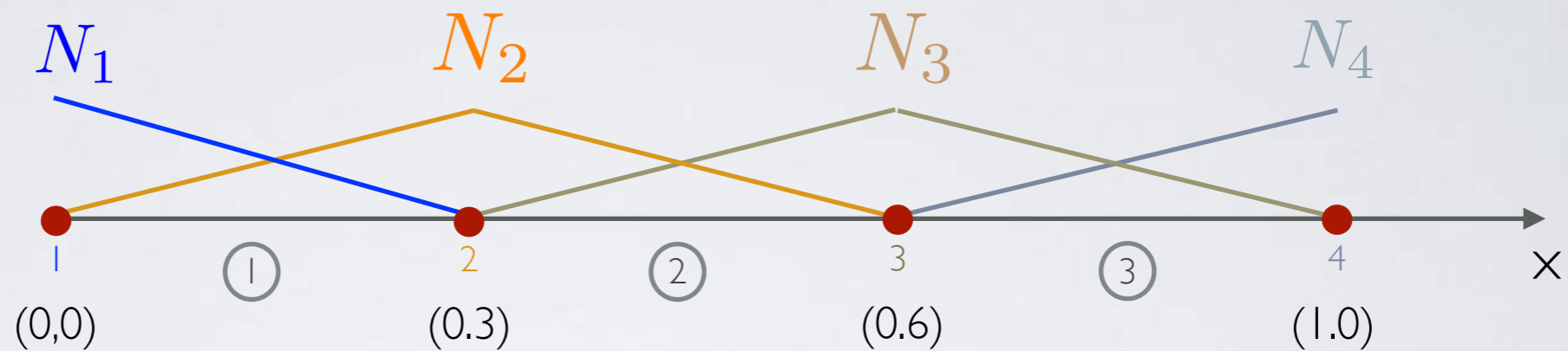
$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

FORMA MATRICIAL

domain
and shape
functions:



element ②
 $\Omega_2^e = [1/3, 2/3]$

$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

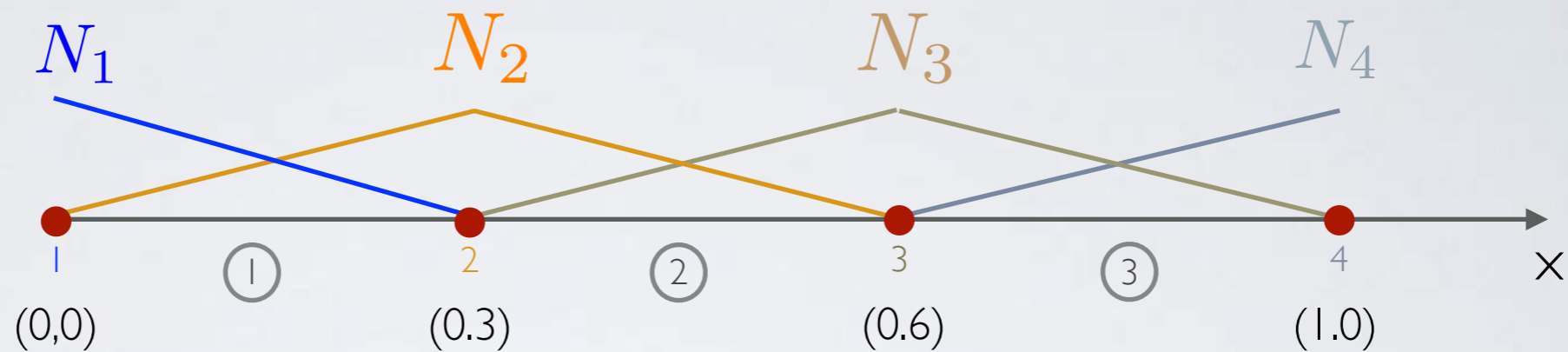
$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$



FORMA MATRICIAL

domain
and shape
functions:



element $\textcircled{3}$
 $\Omega_3^e = [2/3, 1]$

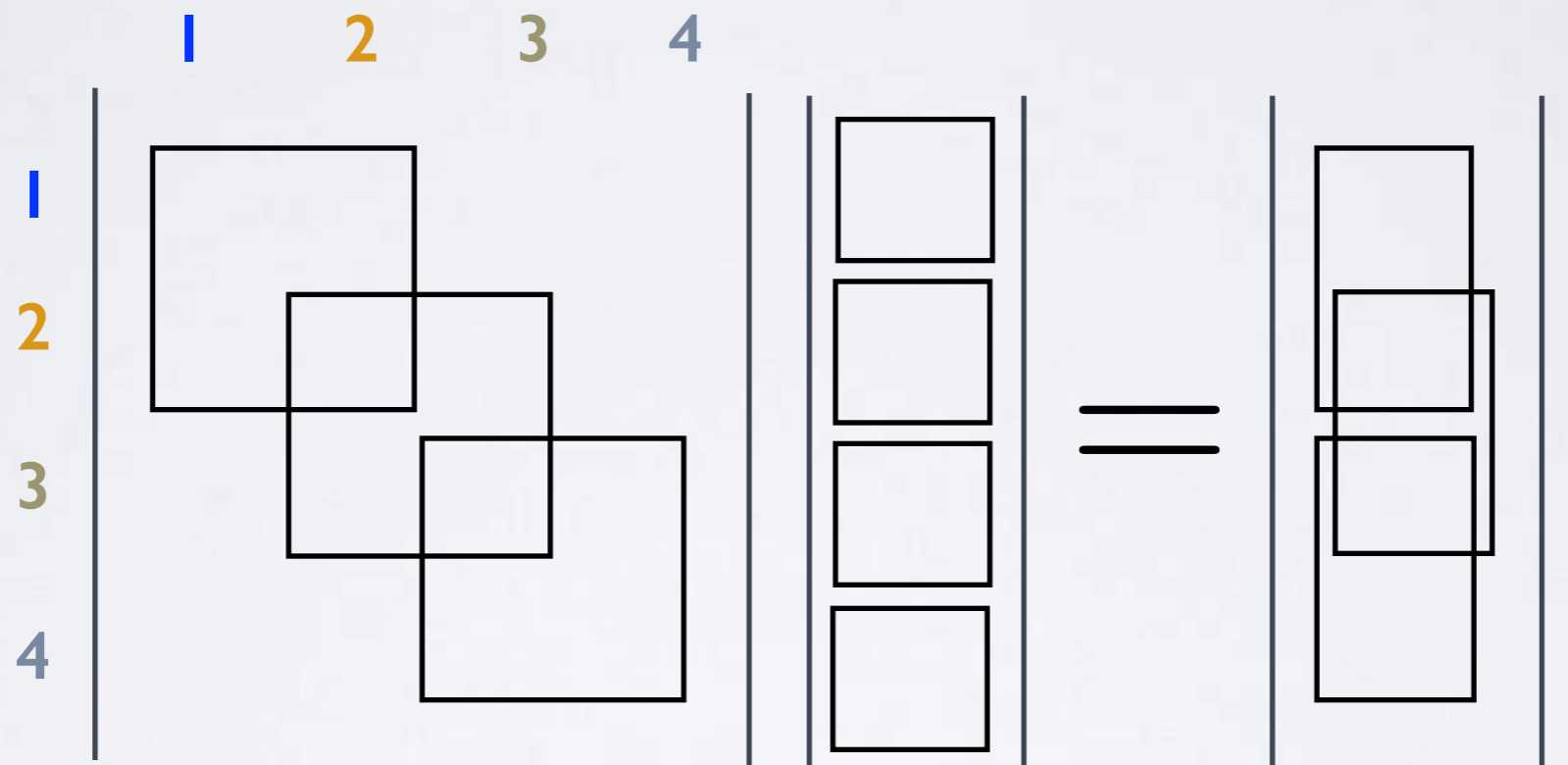
$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 + \text{b.c. at } x = 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 + 1 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$



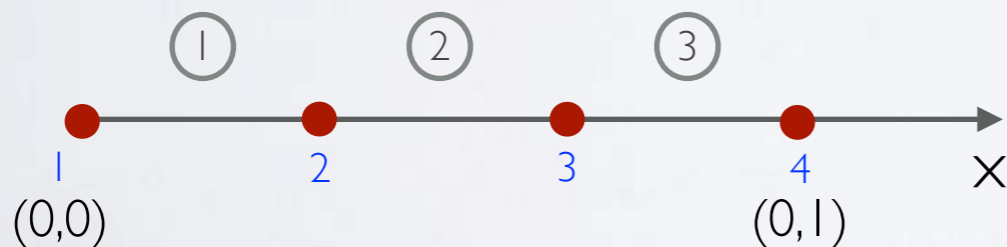
$K_{ij} - M_{ij}$ u_i $b_i + b.c.$

PROBLEMA 1D - EXERCÍCIO

Encontrar u no domínio $\Omega = [0, 1]$ tal que:

$$\frac{d^2 u}{dx^2} + u + 1 = 0 \quad \left| \quad \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = -u \end{array} \right. \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \begin{array}{l} \text{condição de} \\ \text{contorno} \end{array}$$

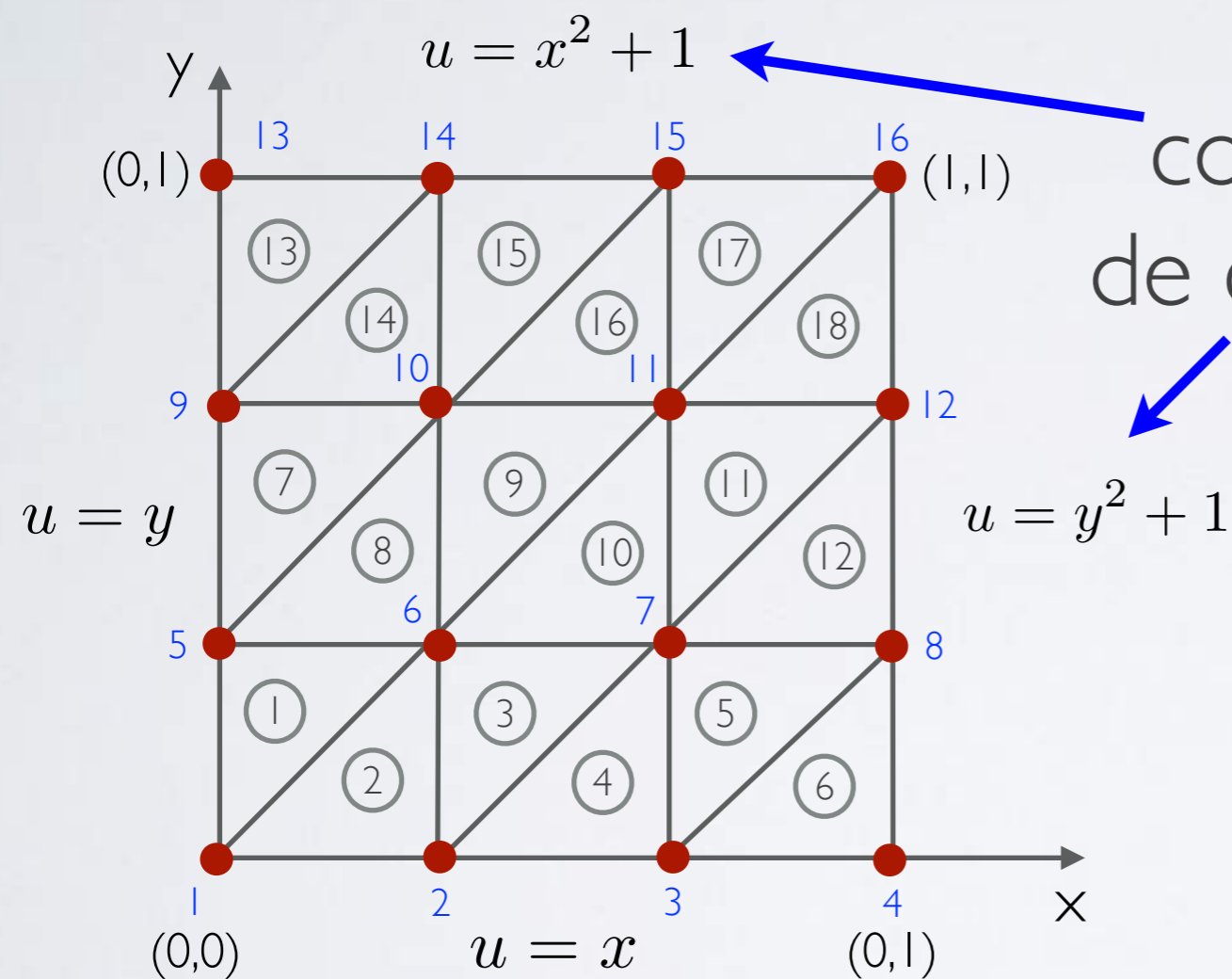
domínio: $h_1 = h_2 = h_3 = 1/3$



Solução: $u_2 = 0.251$;
 $u_3 = 0.363$;
 $u_4 = 0.328$

PROBLEMA 2D - EXERCÍCIO

Encontrar u no domínio $\Omega = [0, 1] \times [0, 1]$ tal que:



Equação:

$$\nabla^2 u = 0$$

Solução: $u_6 = 0.611$;
 $u_7 = 0.889$;
 $u_{10} = 0.889$;
 $u_{11} = 1.167$

PROBLEMA COMPUTACIONAL



Implementar numericamente usando linguagem de programação de sua preferência (Python, Matlab, C/C++, fortran, java) para os casos 1D e 2D apresentados anteriormente:

- 1) para qualquer número de elementos;
- 2) utilizar elementos quadráticos 1D;
- 3) para o caso 2D, usando malha não-estruturada.
- 4) para o caso 2D, acrescentar a derivada temporal, considerando $u=0$ para os pontos no tempo inicial.